Are there Thermodynamical Degrees of Freedom of Gravitation?

H.-H. v. Borzeszkowski and T. Chrobok
Institut für Theoretische Physik, Technische Universität Berlin, Hardenbergstr. 36, D-10623 Berlin, Germany

In discussing fundamentals of general-relativistic irreversible continuum thermodynamics, this theory is shown to be characterized by the feature that no thermodynamical degrees of freedom are ascribed to gravitation. However, accepting that black hole thermodynamics seems to oppose this harmlessness of gravitation one is called on consider other approaches. Therefore, in brief some gravitational and thermodynamical alternatives are reviewed.

I. INTRODUCTION

Due to the equivalence of energy, inertia and gravitation, the gravitational fields are universally coupled to all physical matter and systems, respectively. As a consequence, gravitation cannot be switched off or screened such that one has always to regard as well the influence of external gravitational fields as of the gravitational self-field of the system under consideration. Thus, the laws of thermodynamics and general relativity theory must be applied simultaneously. And to ensure that this provides a self-consistent physical description one has to unify or, at least, to harmonize these laws. The fact that gravitational fields are generally so weak that one can neglect their action does not free oneself from this task since, first, one needs its solution for an exact description and, second, in cosmology and astrophysical objects like neutron stars the gravitational field is strong.

The standard version of general-relativistic irreversible thermodynamics results as an adaption of the non-relativistic continuum theory of irreversible processes near the equilibrium (non-relativistic TIP) to general relativity theory (GRT); therefore, let us call it general-relativistic TIP. It is reached by following that rule which is given by the above-mentioned principle of equivalence: First one has to look for a special-relativistic version of the theory under consideration (this provides Lorentz-covariant basic laws formulated in the Minkowski space-time), afterwards one has to transit to the general-relativistic formulation with coordinate-covariant laws formulated in a Riemannian space-time, whose curvature tensor measures the gravitational field.

This straight road from non-relativistic TIP to general-relativistic TIP does not seem to encounter any hindrance. Our point, however, is that the theory established in this manner only leads to a harmonization of irreversible thermodynamics and GRT. It retains its physical justification only by presupposing that the gravitational field has no thermodynamical degrees of freedom. However, this assumption is questionable for two reasons: (i) The resulting general-relativistic thermodynamics misses most of that content which it started from and which legitimizes it as thermodynamics. Its general-relativistic generalization made TIP a physically void scheme. This loss concerns the first law of thermodynamics and the hypothesis of local equilibrium (and, thus, also the second law of thermodynamics). (ii) The thermodynamics of black holes seems to show that gravitation has thermodynamical degrees of freedom.

Therefore, general-relativistic TIP (and, as will be shown, certain of its generalizations) has to be reconsidered in order to win hints at a theory representing a genuine unification of irreversible thermodynamics and gravitational theory. As a result of this, all speaks for a thermodynamics far from the equilibrium or/and a relativistic theory of gravitation which, in contrast to GRT, maintains the notion of energy[1].

Of course, thermodynamicists and relativists know a variety of arguments in favor of a modification of their respective theories, TIP and GRT. Even more, there are conceptions for such modifications and also elaborated alternative theories. Thus, the purpose of this paper is to add a further argument stemming from a critical investigation of general-relativistic TIP.

The paper is organized as follows. To remind of the original physical content of continuum thermodynamics we intend to show as missing in its general-relativistic version, in Sec. II, the transition from non-relativistic TIP to special-relativistic TIP is briefly summarized. In Sec. III, we turn to general-relativistic TIP and some of its extensions, discuss the problems of unification one meets therein, and propose possible solutions of the problem.

[1]Possibly, the unification of thermodynamics and general relativity can only be reached in the statistical frame. But, nevertheless, also the fluid approximation of statistical thermodynamics has to be compatible with GRT.

II. FROM THE NON-RELATIVISTIC FIELD THEORY OF IRREVERSIBLE PROCESSES (NON-RELATIVISTIC TIP) TO ITS SPECIAL-RELATIVISTIC VERSION (SPECIAL-RELATIVISTIC TIP)

Non-relativistic TIP (cf., e.g. [1]) is a field theory at which one arrives by reformulating classical (quasi-static) thermodynamics in two steps: First one reinterprets the basic relations of classical thermodynamics (first and second laws, Carnot-Clausius relation, Gibbs law), originally formulated in the state space, as relations in space and time (for arguments, see [2]). Accordingly, one has to replace thermodynamical potentials by space-time functions and 1-forms (or differentials) of quantities defined in the state space by the material (or substantial, or co-moving) derivative, where $d/dt = \partial/\partial t + v^\mu \partial_\mu$ (small Greek indices run from 1 to 3). This provides the relations (U =internal energy, Q =heat supply, W =work, S =entropy, Θ =temperature):

$$\textit{First law of thermodynamics:} \qquad \frac{dU}{dt} = \frac{dQ}{dt} + \frac{dW}{dt} \tag{1}$$

$$\textit{Second law of thermodynamics:} \qquad \frac{dS}{dt} = \sigma \geq 0 \tag{2}$$

$$\textit{Carnot-Clausius theorem (}^o\textit{ denotes the equilibrium value):} \quad \Theta \frac{d\overset{o}{S}}{dt} = \frac{dQ}{dt} \tag{3}$$

$$\textit{Gibbs law:} \qquad \frac{dU}{dt} = \Theta \frac{dS}{dt} + \frac{dW}{dt} \tag{4}$$

In a second step, these integral relations are rewritten in the form of local laws by introducing densities and fluxes of the extensive quantities. Following this procedure one arrives at non-relativistic TIP which is based on the balances for mechanical quantities and the first and the second laws of thermodynamics. Thus, these laws take the general form [3]:

$$\frac{\partial u_A}{\partial t} + \partial_\mu \phi^\mu_A = r_A \tag{5}$$

where $u_A(x^\mu, t)$ are the wanted fields, the $\phi^\mu_A(x^\mu, t)$ the fluxes, $r_A(x^\mu, t)$ the supply and production terms. They are differential equations for the wanted fields and valid for arbitrary materials. From them one can again return to their integral formulations. These relations must be supplemented by constitutive equations which are defined on the state space and depend on the specific material under consideration. Inserting them into the balance relation one obtains balances on the state space.

For this theory stems from classical thermodynamics it maintains its basic assumption, namely the hypothesis of local equilibrium saying in particular (see, e.g., [3]) that (i) the state space of non-relativistic TIP is the equilibrium state space, (ii) the "volume elements" are considered as systems in local equilibrium ("Schottky systems"), and the values of the fields do not change in a volume element, but differ only from volume element to volume element thus describing a non-equilibrium situation.

This means also that, in non-equilibrium, the time-derivative of the entropy satisfies the Carnot-Clausius theorem and, thus, the Gibbs law ($\rho c^2 = n(mc^2 + \epsilon)$, where ϵ is the internal energy and n the particle density; μ denotes the chemical potential):

$$\frac{d(ns)}{dt} = \frac{1}{\Theta}\frac{d(\rho c^2)}{dt} - \frac{\mu}{\Theta}\frac{dn}{dt} \tag{6}$$

Now this law holds not just for transitions between equilibrium states but for arbitrary infinitesimal displacements. Assuming the Casimir-Onsager relations the Gibbs law together with the balance equations for mass and internal energy enables one to calculate those fluxes and forces which determine the entropy production.

Turning to the special-relativistic generalization of TIP [4–11], one has to replace the above-given basic relations by such ones that are covariant with respect to global Lorentz transformations. Accordingly, the basic quantities have to be Lorentz tensor fields (we choose the signature -2). The intensive quantities temperature Θ, pressure p, chemical potential μ are associated to scalars measured by an observer at rest with respect to the fluid, the extensive quantities

entropy, volume, particle number are associated to vectors S^a, u^a, N^a, and energy and momentum to a (symmetric) tensor T^{ab} (small Latin indices run from 0 to 3). These quantities have to satisfy the relations[2]:

$$\textit{Balance equations:} \qquad \nabla_a N^a = 0 \tag{7}$$

$$\nabla_a T^{ab} = 0 \tag{8}$$

$$\textit{Dissipative equation:} \qquad \nabla_a S^a = \sigma \geq 0 \tag{9}$$

$$\textit{Carnot-Clausius condition:} \qquad S^a = \phi^a - \beta_b T^{ab} - \alpha N^a \tag{10}$$

$$\textit{Gibbs law:} \qquad dS^a = -\alpha dN^a - \beta_b dT^{ab} \tag{11}$$

where u_a is the 4-velocity, $\beta_a = u_a/\Theta$ the inverse temperature vector, $\phi^a = p\beta^a$ the fugacity vector, and $\alpha = \mu/\Theta$. It is assumed that the equilibrium energy-momentum tensor $\overset{o}{T}{}^{ab}$ is given by that one of ideal matter and α and $\beta^2 = \beta_a\beta^a = c^2/\Theta^2$ are equal to their equilibrium values.

In global thermal equilibrium, the thermodynamical gradients are constrained by

$$\partial_a \alpha = 0 \qquad \nabla_a \beta_b + \nabla_b \beta_a = 0 \tag{12}$$

This approach to special-relativistic TIP[3] is characterized by the assumption that off-equilibrium thermodynamics is described by the same variables and with the same entropy current as in equilibrium thermodynamics. This framework can be generalized by assuming a "second-order" scheme where

$$S^a = p(\alpha, \beta)\beta^a - \alpha N^a - \beta_b T^{ab} - Q^a(\delta N^c, \delta T^{cd}, X_A^{c\cdots}) \tag{13}$$

where Q^a is of second order in the derivatives $\delta N^c, \delta T^{cd}$ and generally depends also on other variables $X_A^{c\cdots}$ that vanish in equilibrium (in first-order theory one had $Q^a = 0$). Specific versions of this scheme generally considered in [11] are extended thermodynamics theories [12,14–16] and divergence thermodynamic theories [17–19].[4]

Special-relativistic TIP and its generalizations given by (13) are also thermodynamical theories near the equilibrium based on the Carnot-Clausius relation and the Gibbs law. Similar as in non-relativistic TIP, it is again possible to return to integral balance relations which, for closed systems, become conservation laws.

Regarding volume forces this scheme can be extended. For instance, in the case of electrodynamical forces the above-given basic Minkowski tensors have to be supplemented by the electromagnetic excitation tensor H^{ab} ($H^{ab} = -H^{ba}$), the electromagnetic field strength tensor F_{ab} ($F_{ab} = -F_{ba}$), and the basic relations have to be modified by adding the energy-momentum tensor of the electromagnetic field to T^{ab}. Furthermore, the balance equations must by supplemented by the electromagnetic field equations (containing the electric current vector J^a):

$$\nabla_b H^{ab} = J^a \qquad \nabla_{[a} F_{bc]} = 0 \tag{14}$$

The Carnot-Clausius relation remains unchanged if one assumes that the electromagnetic field does not contribute to the entropy current (cf, e.g., [9]).

III. ON GENERAL-RELATIVISTIC TIP AND SOME OF ITS GENERALIZATIONS

If gravitational fields are incorporated the equivalence principle forbids one to do this in the same way as in the electromagnetical case. Instead, one has to regard gravitational effects by "lifting" all equations from the Minkowski

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[2] In [9], in contrast to eq. (10), the entropy vector is assumed to be $S^a = -\beta_a T^{ab} - \alpha N^a$.

[3] The theories of Eckart [4] and Landau and Lifshitz [5] are typical representatives of this approach. (For a comparison of them, see [12,13].)

[4] Since the specific form of S^a is not relevant for our further arguments we shall not comment its different versions, either in first-order or second-order theories.

into the Riemann space-time. This means to replace in the equations of special-relativistic TIP the 4-metric η_{ab} by the 4-metric g_{ab} and the partial 4-derivative ∇ by the covariant derivative D.

This procedure is unique because the lifting procedure can be performed before the constitutive equations have to be taken into consideration, i.e., on a level where still all relations to be lifted are differential equations of first order. (Problems occur for differential equations of higher-order because the covariant derivatives do not commute so that then one encounters a derivative-ordering problem.)

In general-relativistic TIP, the basic fields are Riemann tensors and the basic relations read as follows:

$$\textit{Balance equations:} \quad D_a N^a = 0 \tag{15}$$

$$D_a T^{ab} = 0 \tag{16}$$

$$\textit{Dissipative equation:} \quad D_a S^a = \sigma \geq 0 \tag{17}$$

$$\textit{Einstein equations:} \quad G_{ab} = \kappa T_{ab}. \tag{18}$$

Furthermore, one requires again the Carnot-Clausius relation (10) and the Gibbs law (11).

A. Problems of general-relativistic TIP

(1) The local balance equations generally cannot be rewritten in an integral form. As a consequence, the local "conservation" laws (first of all, the first law of thermodynamics) are no genuine conservation laws. This is due to the fact that there is no energy-momentum tensor of gravitation [20–23]. Indeed, the Gauß theorem allowing to transit from local divergence equations to (integral) conservation laws holds in the Riemann space only for the divergence of vectors. In the latter case one has

$$D_a A^a = 0 \Rightarrow \int_{V_4} D_a A^a d^4x = 0 \tag{19}$$

$$\int_{V_4} D_a A^a d^4x = \int_{\partial V_4^{(1)}} A_a df^a + \int_{\partial V_4^{(2)}} A_a df^a = 0 \tag{20}$$

(V_4 denotes a 4-dimensional volume, i.e., a 4-cylinder, and ∂V_4 two space-like interfaces of the cylinder) such that one obtains

$$\int_{\partial V_4} A_a df^a = constant \tag{21}$$

saying that the quantity is independent of the interface ∂V_4. But for tensors of higher rank, e.g., for tensors of second rank one has

$$D_a A^{ab} = 0 \Rightarrow \int_{V_4} D_a A^{ab} d^4x = 0 \tag{22}$$

which, due to a missing Gauß theorem, cannot be rewritten in a form similar to (20). This implies that eq. (16) is no conservation law [20–23]. The missing energy-momentum law is due to the above-mentioned fact that there does not exist an energy-momentum tensor of gravitational fields. This becomes obvious when one writes the latter relation (16) equivalently as

$$D_a T^{ab} = \nabla_a (T^{ab} + t^{ab}) = 0 \tag{23}$$

Thus, the covariant divergence is transposed into an ordinary divergence what enables one to apply the Gauß theorem. But it does not help because t^{ab} is no tensor; it is the so-called gravitational energy-momentum complex. Only in equilibrium, where, due to (12), one has a time-like Killing vector field β_a eq. (16) becomes the energy conservation law.

As an implication of the missing law of energy- momentum conservation, there is no first law in general-relativistic thermodynamics. In special-relativistic thermodynamics, one obtains this law by multiplying the law of energy-momentum conservation (8) with the velocity u_b (for details, see [13]):

$$u_b\nabla_a T^{ab} = 0 \tag{24}$$

With the expression of the energy-momentum tensor,

$$T^{ab} = c^{-2}\rho u^a u^b + c^{-2}(q^a u^b + q^b u^a) + p h^{ab} + \pi^{ab} \tag{25}$$

where

$$\rho: = c^{-2}u_a u_b T^{ab} \qquad \text{(energy density)} \tag{26}$$
$$q^a: = h^a{}_c T^{bc} u_b = I^a + h h^{ab} N_b \qquad \text{(heat flow)} \tag{27}$$
$$P^{ab}: = h^a{}_c T^{cd} h^b{}_d = p h^{ab} + \pi^{ab} \qquad \text{(pressure tensor)} \tag{28}$$
$$h^{ab}: = \eta^{ab} - c^{-2}u^a u^b \qquad \text{(projector orthogonal to } u_a\text{)} \tag{29}$$
$$h: = \frac{\rho + p}{n} \qquad \text{(specific enthalpy)} \tag{30}$$

Eq. (24) provides the first law in the form ($\nabla = N^a\nabla_a$):

$$\nabla e + p\nabla n^{-1} = \pi^{ab}\nabla_b u_a - \nabla_a I^a + (hn)^{-1} I^a h_{ab}\nabla^b p \tag{31}$$

This law shows that the change of energy e is produced by two work and two heat terms.

In general-relativistic thermodynamics, instead of (8), one has (16). Therefore, one obtains instead of Eq. (31) the relation ($D := N^a D_a$):

$$De + pDn^{-1} = \pi^{ab} D_b u_a - D_a I^a + (hn)^{-1} I^a h_{ab} D^b p \tag{32}$$

Because of the missing physical interpretation of eq. (16) as a local energy-momentum-conservation, it cannot be interpreted as first law in the original spirit of thermodynamics, but as one of the equations of motion.

Now one could oppose that this is nothing but the well-known problem that in GRT there is no law of energy conservation and that one has to live with it in all branches of GRT. Why not in general-relativistic thermodynamics, either? Our point, however, is that here, because the energy conservation is a basic law of thermodynamics, this problem is more serious than in other branches of physics. If, for instance, one considers the influence of gravitation on a Dirac field then one has also to lift the Dirac equations in the Riemann space-time. Of course, then the notion of energy loses its meaning, too. However, in contrast to general-relativistic TIP and some of the above-mentioned extensions, by this one does not lose basic laws determining the dynamics of the physical system.

(2) The loss of the notion of energy leads to problems with the entropy, too. It is true that the dissipative relation (17) contains the divergence of a vector such that the reformulation described in Eqs. (19)-(21) is possible, but without a notion of energy, in the framework of TIP, the notion of entropy is physically meaningless.

(3) The basic assumption of TIP, namely the hypothesis of local equilibrium (including Carnot-Clausius relation and the Gibbs law), makes no sense in GRT since there are no Schottky systems.

(4) One meets problems with the interpretation of the metric and Einstein's equations that can be summarized by the following questions (for a discussion of the first two questions, see also [24]). (i) Is the metric an independent variable of the state space, or is it given by a constitutive equation? (ii) What about Einstein's equations? They must be solved simultaneously with the balances. But are they balance equations? (iii) The Einstein equations imply $D_a T^{ab} = 0$. Therefore, one has to ask whether in GRT room is left for a genuine first law.

Usually general-relativistic TIP and its extensions are justified by considering such volume cells $d\Sigma_b$ (arbitrarily oriented elements of 3-areas) that the integral of the 4-momentum flux $dp^a = c^{-1}T^{ab}d\Sigma_b$ should vanish to an order higher than the 4-volume, when evaluated in locally inertial coordinates (where $\partial_a g_{bc} = 0$) [11]. Then all relations reduce to their special-relativistic versions which can be interpreted in the original spirit of thermodynamics. In particular, one has infinitesimal Schottky systems that allow one to define local equilibrium. That means, the above-discussed general-relativistic thermodynamics is based on the assumption that the gravitational field g_{ab} has no thermodynamical degrees of freedom. This assumption harmonizes but does not unify both theories. However, this seems to contradict black hole thermodynamics (see, e.g., [25] and the literature therein) showing that the gravitational field can be ascribed an entropy.

B. Some possible way outs of the dilemma

(1) One maintains general-relativistic TIP and considers thermodynamical problems only for special solutions having a time-like Killing vector or a conformal Killing vector field. In these cases one can define energy and formulate a

corresponding conservation law. For instance, in Friedman-Robertson-Walker universes exist time-like conformal Killing vector fields what allows one to establish cosmological thermodynamics.

(2) One maintains GRT and replaces non-relativistic TIP by a non-equilibrium thermodynamics requiring neither conservation laws nor the hypothesis of local equilibrium.

(3) Alternatively to (2), one maintains non-relativistic TIP and replace GRT by a generalized GRT providing conservation laws (e.g., the teleparallized GRT [26,27]) and unify both fields.

(4) One combines the modifications of TIP and GRT.

(5) Under relativists it is a popular idea to consider black hole thermodynamics as guideline to a general-relativistic thermodynamics (see, e.g., [25]). This idea is interesting for it enables one, at least for some special solutions of Einstein's equations, to geometrize thermodynamical quantities like temperature and entropy. As mentioned above, in contrast to general-relativistic TIP, where these quantities are only ascribed to the system moving in the gravitational field but not to this field itself, here thermodynamical quantities are ascribed to the gravitational field, too. However, it is difficult to see how black hole thermodynamics can be connected with ordinary thermodynamics. In [28] it is attempted to build a bridge between them by deriving a "universal parameter thermodynamics for rotating fluids".

(6) Another possibility is to incorporate the entropy as a primary quantity given by the gravitational field. This idea goes back to Penrose [29] who recognized that the Weyl curvature should vanish at the initial singularity (this is sometimes called *Weyl curvature conjecture*). Following this idea several measures for the gravitational entropy were formulated (see, e.g., [30] and the references therein). All these measures are build from curvature invariants such that for vanishing Weyl curvature the gravitational entropy also vanishes. The advantage of this attempt is that the gravitational field obtains thermodynamical degrees of freedom, but the problem of the missing first law still remains open.

(7) Moreover, one may find a formulation of Einsteins equations which, at least partly, has the structure of divergence equations. These equations can be included in the scheme of divergence thermodynamic theories. Physically this would mean to ascribe the gravitational field thermodynamical degrees of freedom (a paper concerning this topic is in preparation).

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